Vector Decomposition and Modeling

To understand projectile motion, we decompose vectors into horizontal and vertical components using Cartesian coordinates. This allows us to construct a two-dimensional model of motion that aligns with the results of dimensional analysis.

1. Dimensional Analysis of the Horizontal Range (*R*)

We begin by modeling the horizontal displacement (*R*) of a projectile fired horizontally from a height (*h*) with initial speed (v_0). The acceleration due to gravity is (*g*).

We hypothesize the relationship:

$$R = C.v_0^a.h^b.g^c$$

Step 1: Dimensional representation

In dimensional analysis, physical quantities are expressed in terms of fundamental dimensions to ensure consistency across equations. The fundamental dimensions typically used are:

- L – length
- T – time
- M – mass (not used in this case)

Each variable in our model is assigned a dimensional formula based on how it behaves physically:

- [R]=L ⟶ Horizontal range is a length.
- $[v_0]=LT^{-1}$ ⟶ Velocity is length per time.
- [h]=L ⟶ Height is a length.
- $[g]=LT^{-2}$ ⟶ Gravitational acceleration is length per time squared.

We substitute these into the hypothesized equation: $R = C.v_0^a.h^b.g^c$

Using dimensions:

$$L^{-1}T^{0} = \left(LT^{-1}\right)^{a}.(L)^{b}.\left(LT^{-2}\right)^{c}$$

Combining exponents:

$$L^{-1}T^{0} = (L)^{a+b+c}.(T)^{-a-2c}$$

Equating powers:

For L: $a + b + c = 1$

For T: $-a - 2c = 0$

Solving:

From the time equation: : $a = -2c$

Substituting into the length equation:

$$(-2c) + b + c = 1$$
$$b - c = 1$$
$$b = c + 1$$

As discussed previously, dimensional analysis provides a system of two equations with three unknowns, meaning there isn't a unique solution without further physical insight. The true physical relationship requires specific exponents.

Assume $c = -\frac{1}{2}$ (which aligns with the known physical relationship derived from kinematics):

If $c = -\frac{1}{2}$:

$a = -2\left(-\frac{1}{2}\right) = 1$

$b = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$

So, the functional relationship becomes:

$$R = C.v_0^{1}.h^{\frac{1}{2}}.g^{-\frac{1}{2}} \quad \Rightarrow \quad R = C.v_0.\sqrt{\frac{h}{g}}$$

This form is consistent with the true physical relationship, where $C = 2$

2. Vector Decomposition and Modeling

To understand projectile motion, we decompose vectors into horizontal and vertical components using Cartesian coordinates. This allows us to construct a two-dimensional model of motion that aligns with the results of dimensional analysis.

You've defined the key vectors:

- Initial velocity vector: $v_0 = \langle v_0, 0 \rangle$ (assuming horizontal launch from $x = 0, y = h,$ so y component of initial velocity is 0)
- Acceleration vector: $a=\langle 0,-g \rangle$ (gravity acts only in the negative y direction)
- Displacement vector: $d=\langle R,-h \rangle$ (where R is the horizontal range and $-h$ is the vertical displacement from the initial height to the ground, assuming initial height h)

The fundamental kinematic equation for displacement under constant acceleration is:

$$d = v_0 . t + \frac{1}{2} . a . t^2$$

Let's substitute the component forms of the vectors into this equation:

$$\langle R, - h \rangle = \langle v_0, 0 \rangle . t + \frac{1}{2} \langle 0, - g \rangle t^2$$

Now, let's perform the vector operations component by component:

1. **Multiply by scalar t for the initial velocity term:**

$$\langle v_0, 0 \rangle . t = \langle v_0 t, 0 . t \rangle = \langle v_0 t, 0 \rangle$$

2. **Multiply by scalar $\frac{1}{2}.t^2$ for the acceleration term:**

$$\frac{1}{2}\langle 0, -g\rangle t^2 = \langle \frac{1}{2}.0.t^2, \frac{1}{2}(-g).t^2\rangle = \langle 0, -\frac{1}{2}g.t^2\rangle$$

3. **Add the resulting vectors:**

$$\langle R, -h\rangle = \langle v_0 t, 0\rangle.t + \langle 0, -\frac{1}{2}g.t^2\rangle$$

$$\langle R, -h\rangle = \langle v_0 t + 0\ ,\ 0 - \frac{1}{2}g.t^2\rangle$$

$$\langle R, -h\rangle = \langle v_0 t\ ,\ -\frac{1}{2}g.t^2\rangle$$

4. **Equate the components:**

From the horizontal (x) components:

$$R = v_0.t$$

From the vertical (y) components:

$$-h = -\frac{1}{2}.g.t^2 \quad \Rightarrow \quad h = \frac{1}{2}.g.t^2$$

From the vertical equation, we can solve for time t:

$$t^2 = \frac{2h}{g} \implies t = \sqrt{\frac{2h}{g}}$$

(We take the positive root for time).

Finally, substitute this expression for t back into the horizontal range equation:

$$R = v_0.\left(\sqrt{\frac{2h}{g}}\right) \implies R = v_0.\sqrt{\frac{2h}{g}}$$

This vector decomposition perfectly demonstrates and derives the relationships for the time of fall and the horizontal range, confirming the independence of the horizontal and vertical

motions and aligning with the dimensional model you presented.

3. Functional Modeling and Graph Interpretation

A table of data was constructed using realistic values for (h), (v_0), and (g), then used to graph:

Sample Values:

Trial	v_0 (m/s)	h (m)	g (m/s)	R (m)
1	4.0	2.0	9.8	2.55
2	5.0	2.0	9.8	3.19
3	6.0	2.0	9.8	3.83
4	5.0	3.0	9.8	3.91

From this table, students graph (R) vs. (v_0), (R) vs. (h), and explore inverse square root relations with (g).

Let's use the provided sample data to explore these relationships, based on the theoretical relationship: $R = v_0 \cdot \sqrt{\frac{h}{g}}$

This relationship tells us:

1. R is directly proportional to v_0 (when h and g are constant).
2. R is directly proportional to $\sqrt{h}$ (when v_0 and g are constant).
3. R is inversely proportional to $\sqrt{g}$ (when v_0 and h are constant).

Step 1: Analyze the relationship between R and v_0 (R vs. v_0 graph)

To do this, we need to find trials where h and g are kept constant, and only v_0 changes.

- **Relevant Trials:** Trials 1, 2, and 3.

  Trial 1: v_0 =4.0 m/s, h=2.0 m, g=9.8 m/s , R=2.55 m

  Trial 2: v_0 =5.0 m/s, h=2.0 m, g=9.8 m/s , R=3.19 m

  Trial 3: v_0 =6.0 m/s, h=2.0 m, g=9.8 m/s , R=3.83 m

- **Hypothesis:** Based on : $R = v_0.\sqrt{\frac{h}{g}}$, if h and g are constant, then R should be directly proportional to v_0 . This means a graph of R vs. v_0 should be a straight line passing through the origin.

- **Calculations/Check:** Let's calculate the ratio R/v_0 for these trials to see if it's roughly constant.

  Trial 1: R/v_0 = 2.55/4.0 = 0.6375

  Trial 2: R/v_0 = 3.19/5.0 = 0.638

  Trial 3: R/v_0 = 3.83/6.0 = 0.6383

  The ratios are very close, indicating a direct linear relationship.

- **Graph Interpretation:**

  X-axis: v_0 (m/s)

  Y-axis: R (m)

  Expected Shape: A straight line passing through the origin (0,0).

  Slope: The slope of this line would be $\sqrt{\frac{2h}{g}}$. For these trials,

$$\sqrt{\frac{2 \cdot 2.0}{9.8}} = \sqrt{\frac{4.0}{9.8}} \approx \quad \sqrt{0.408} \approx 0.639.$$

This matches our calculated R/v_0 values, confirming the relationship.

Step 2: Analyze the relationship between R and h (R vs. h graph)

To do this, we need to find trials where v_0 and g are kept constant, and only h changes.

- **Relevant Trials:** Trials 2 and 4.

  Trial 2: v_0 = 5.0 m/s, h=2.0 m, g=9.8 m/s , R=3.19 m

  Trial 4: v_0 =5.0 m/s, h=3.0 m, g=9.8 m/s , R=3.91 m

- **Hypothesis:** Based on $R = v_0 . \sqrt{\frac{h}{g}}$, , if v_0 and g are constant, then R should be directly proportional to $\sqrt{h}$. This means a graph of R vs. h should be a straight line passing through the origin.

- **Calculations/Check:** Let's calculate $\sqrt{h}$ and then the ratio R/$\sqrt{h}$ for these trials.

  Trial 2: $\sqrt{h}$ =2.0 ≈ 1.414

  R/$\sqrt{h}$ = 3.19/1.414 ≈ 2.256

  Trial 4: $\sqrt{h}$ =3.0 ≈ 1.732

  R/$\sqrt{h}$ =3.91/1.732 ≈ 2.257

  The ratios are very close, indicating a direct linear relationship between R and $\sqrt{h}$.

- **Graph Interpretation:**

  X-axis: $\sqrt{h}$ (m)

  Y-axis: R (m)

Expected Shape: A straight line passing through the origin (0,0).

Slope: The slope of this line would be $v_0 \cdot \sqrt{\frac{2}{g}}$, . For these trials,

$$5.0 \cdot \sqrt{\frac{2}{9.8}} = 5.0 \cdot \quad \sqrt{0.204} \approx 5.0 \cdot 0.451 \approx 2.255.$$

This matches our calculated R/$\sqrt{h}$ values, confirming the relationship.

Step 3: Explore inverse square root relations with g

We don't have enough data points where v_0 and h are constant, and g varies to perform a robust graph analysis from the provided table. However, we can explain the theoretical expectation.

- **Hypothesis:** Based on $R = v_0 \cdot \sqrt{\frac{h}{g}}$, , if v_0 and h are constant, then R is inversely proportional to$\sqrt{g}$. This means:

  R∝1/$\sqrt{g}$

  R=constant· $g^{-0.5}$

- **Graphing Strategy:**

  To linearize this relationship, students would typically plot R vs. 1/$\sqrt{g}$.

  X-axis: 1/$\sqrt{g}$ (s/m)

  Y-axis: R (m)

  Expected Shape: A straight line passing through the origin (0,0).

  Slope: The slope of this line would be $v_0\sqrt{2h}$.

Summary of Functional Modeling and Graph Interpretation:

By strategically selecting data points where only one variable is changed, we can isolate its

effect on the range R.

- **R vs.** v_0 **(constant h,g):** Plots as a straight line through the origin, indicating R$\propto v_0$.
- **R vs.** $\sqrt{h}$**. (constant** v_0**,g):** Plots as a straight line through the origin, indicating R $\propto \sqrt{h}$.
- **R vs. 1/**$\sqrt{g}$ **(constant** v_0**,h):** Plots as a straight line through the origin, indicating R $\propto$ 1/$\sqrt{g}$.

These graphical interpretations provide strong experimental evidence supporting the theoretically derived functional form: on $R = v_0.\sqrt{\frac{h}{g}}$. Students would perform these graphs, observe the linearity, and potentially calculate the slopes to verify the constant terms ($\sqrt{\frac{2h}{g}}$, $v_0.\sqrt{\frac{2}{g}}$, and $v_0.\sqrt{2h}$ respectively) from their experimental data against the known values of v_0 ,h,g.